HOMOLOGICAL DIMENSIONS OF THE AMALGAMATED DUPLICATION OF A RING ALONG A PURE IDEAL

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ABSTRACT. The aim of this paper is to study the classical global and weak dimensions of the amalgamated duplication of a ring R along a pure ideal I.

1. INTRODUCTION

Throughout this paper all rings are commutative with identity element and all modules are unitary.
Let R be a ring, and let M be an R-module. As usual we use $\mathrm{pd}_R(M)$, $\mathrm{id}_R(M)$ and $\mathrm{fd}_R(M)$ to denote, respectively, the classical projective dimension, injective dimension and flat dimension of M. We use also $gldim(R)$ and $wdim(R)$ to denote, respectively, the classical global and weak dimension of R.

For a nonnegative integer n, an R-module M is n-presented if there is an exact sequence $F_n \to F_{n-1} \to ... \to F_0 \to M \to 0$ in which each F_i is a finitely generated free R-module. In particular, "0-presented" means finitely generated and "1-presented" means finitely presented. Set $\lambda_R(M) = \{n/M$ is n-presented $\}$ except if M is not finitely generated. In this last case, we set $\lambda_R(M) = -1$. Not that $\lambda_R(M) \geq n$ is a way to express the fact that M is n-presented.
Given nonnegative integers n and d, a ring R is called an (n, d)-ring if every n-presented R-module has projective dimension $\leq d$, and R is called a weak (n, d)-ring if every n-presented cyclic R-module has projective dimension $\leq d$. For instance, the $(0, 1)$-domains are the Dedekind domains, the $(1, 1)$-domains are the Prüfer domains and the $(1, 0)$-rings are the Von Neumann regular rings (see [1, 11, 12, 13, 14]). A commutative ring is called an n-Von Neumann regular ring if it is an $(n, 0)$-ring. Thus, the 1-von Neumann regular rings are the von Neumann regular rings ([1, Theorem 1.3]) .

The amalgamated duplication of a ring R along an ideal I is a ring that is defined as the following subring with unit element $(1, 1)$ of $R \times R$:

$$R \bowtie I = \{(r, r + i)/r \in R, i \in I\}$$

This construction has been studied, in the general case, and from the different point of view of pullbacks, by D'Anna and Fontana [6]. Also, in [5], they have considered the case of the amalgamated duplication of a ring, in not necessarily Noetherian setting, along a multiplicative canonical ideal in the sense of [10]. In [4], D'Anna has studied some properties of $R \bowtie I$, in order to construct reduced Gorenstein

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rings associated to Cohen-Macaulay rings and has applied this construction to curve singularities. On the other hand, Maimani and Yassemi, in [16], have studied the diameter and girth of the zero-divisor of the ring $R \bowtie I$. Recently in [3], the authors study some homological properties of the rings $R \bowtie I$. Some references are [4, 5, 6, 16].

Let M be an R-module, the idealization $R \propto M$ (also called the trivial extension), introduced by Nagata in 1956 (cf [17]) is defined as the R-module $R \oplus M$ with multiplication defined by $(r, m)(s, n) = (rs, rn + sm)$ (see [7, 9, 11, 12]).

When $I^2 = 0$, the new construction $R \bowtie I$ coincides with the idealization $R \propto I$. One main difference of this construction, with respect to idealization is that the ring $R \bowtie I$ can be a reduced ring (and, in fact, it is always reduced if R is a domain). The first purpose of this paper is to study the classical global and weak dimension of the amalgamated duplication of a ring R along pure ideal R. Namely, we prove that if I is a pure ideal of R, then $wdim(R \bowtie I) = wdim(R)$. Also, we prove that if R is a coherent ring and I is a finitely generated pure ideal of R, then $R \bowtie I$ is an $(1, d)$-ring provided the local ring R_M is an $(1, d)$-ring for every maximal ideal M of R. Finally, we give several examples of rings which are not weak (n, d)-rings (and so not (n, d)-rings) for each positive integers n and d.

2. MAIN RESULTS

Let R be a commutative ring with identity element 1 and let I be an ideal of R. We define $R \bowtie I = \{(r, s)/r, s \in R, s - r \in I\}$. It is easy to check that $R \bowtie I$ is a subring with unit element $(1, 1)$, of $R \times R$ (with the usual componentwise operations) and that $R \bowtie I = \{(r, r + i)/r \in R, i \in I\}$.

It is easy to see that, if π_i $(i = 1, 2)$ are the projections of $R \times R$ on R, then $\pi_i(R \bowtie I) = R$ and hence if $O_i = ker(\pi_i \backslash R \bowtie I)$, then $R \bowtie I/O_i \cong R$. Moreover $O_1 = \{(0, i), i \in I\}$, $O_2 = \{(i, 0), i \in I\}$ and $O_1 \cap O_2 = (0)$.

Our first main result in this paper is given by the following Theorem:

Theorem 2.1. *Let R be a ring and I be a pure ideal of R. Then, $wdim(R \bowtie I) = wdim(R)$.*

To prove this Theorem we need some results.

Lemma 2.2. [4, Proposition 7] *Let R be a ring and let I be an ideal of R. Let P be a prime ideal of R and set:*

- $P_0 = \{(p, p + i)/p \in P, i \in I \cap P\}$,
- $P_1 = \{(p, p + i)/p \in P, i \in I\}$, *and*
- $P_2 = \{(p + i, p)/p \in P, i \in I\}$.

(1) If $I \subseteq P$, then $P_0 = P_1 = P_2$ and $(R \bowtie I)_{P_0} \cong R_P \bowtie I_P$.
(2) If $I \nsubseteq P$, then $P_1 \neq P_2$, $P_1 \cap P_2 = P_0$ and $(R \bowtie I)_{P_1} \cong R_P \cong (R \bowtie I)_{P_2}$.

Lemma 2.3. *Let I be a non-zero flat ideal of a ring R. For any R-module M we have:*

(1) $fd_R(M) = fd_{R \bowtie I}(M \otimes_R (R \bowtie I))$.
(2) $pd_R(M) = pd_{R \bowtie I}(M \otimes_R (R \bowtie I)$.

Proof. Note that the R-module $R \bowtie I$ is faithfully flat since I is flat.
Firstly suppose that $fd_R(M) \leq n$ (resp., $pd_R(M) \leq n$) and pick an n-step flat (resp., projective) resolution of M over R as follows:

$$(*) \quad 0 \to F_n \to F_{n-1} \to \ldots \to F_0 \to M \to 0.$$

Applying the functor $- \otimes_R R \bowtie I$ to $(*)$, we obtain the exact sequence of $(R \bowtie I)$-modules:

$$0 \to F_n \otimes_R (R \bowtie I) \to F_{n-1} \otimes_R (R \bowtie I) \to \ldots \to F_0 \otimes_R (R \bowtie I) \to M \otimes_R (R \bowtie I) \to 0$$

Thus $fd_{R \bowtie I}(M \otimes_R (R \bowtie I)) \leq n$ (resp., $pd_{R \bowtie I}(M \otimes_R (R \bowtie I)) \leq n$).

Conversely, suppose that $fd_{R \bowtie I}(M \otimes_R (R \bowtie I)) \leq n$ (resp., $pd_{R \bowtie I}(M \otimes_R (R \bowtie I)) \leq n$. Inspecting [2, page 118] and since $Tor_R^k(M, R \bowtie I) = 0$ for each $k \geq 1$, we conclude that for any R-module N and each $k \geq 1$ we have:

(1): $Tor_R^k(M, N \otimes_R (R \bowtie I)) \cong Tor_{R \bowtie I}^k(M \otimes_R (R \bowtie I), N \otimes_R (R \bowtie I))$
(2): $Ext_R^k(M, N \otimes_R (R \bowtie I)) \cong Ext_{R \bowtie I}^k(M \otimes_R (R \bowtie I), N \otimes_R (R \bowtie I))$

On the other hand $Tor_R^k(M, N)$ and $Ext_R^k(M, N)$ are direct summands of $Tor_R^k(M, N \otimes_R (R \bowtie I))$ and $Ext_R^k(M, N \otimes_R (R \bowtie I))$ respectively. Then, we conclude that $fd_R(M) \leq n$ (resp., $pd_R(M) \leq n$) and this finish the proof of this result. □

One direct consequence of this Lemma is:

Corollary 2.4. *Let I be a non-zero flat ideal of a ring R. Then:*

(1) $wdim(R) \leq wdim(R \bowtie I)$.
(2) $gldim(R) \leq gldim(R \ltimes I)$.

Proof of Theorem 2.1. The inequality $wdim(R) \leq wdim(R \bowtie I)$ holds directly from Corollary 2.4 since I is pure and then flat. So, only the other inequality need a proof.
Using [7, Theorem 1.3.14] we have:

$$(\top) \quad wdim(R \bowtie I) = sup\{wdim((R \bowtie I)_M) | M \text{ is a maximal ideal of } R \bowtie I\}.$$

Let M be an arbitrary maximal ideal of $R \bowtie I$ and set $m := M \cap R$. Then necessarily $M \in \{M_1, M_2\}$ where $M_1 = \{(r, r+i)/r \in m, i \in I\}$ and $M_2 = \{(r+i, r)/r \in m, i \in I\}$ (by [6, Theorem 3.5]). On the other hand, $I_m \in \{0, R_m\}$ since I is pure and m is maximal in R (by [7, Theorem 1.2.15]). Then, testing all cases of Lemma 2.3, we resume two cases;

(1) $(R \bowtie I)_M \cong R_m$ if $I_m = 0$ or $I \nsubseteq m$.
(2) $(R \bowtie I)_M \cong R_m \times R_m$ if $I_m = R_m$ or $I \subseteq m$.

Hence, we have $wdim((R \bowtie I)_M) = wdim(R_m) \leq wdim(R)$. So, the desired inequality follows from the equality $(\top)$. □

Corollary 2.5. *Let I be a finitely generated pure ideal of a ring R. Then R is a semihereditary ring if, and only if, $R \bowtie I$ is a semihereditary ring.*

Proof. Follows immediately from Theorem 2.1 and [3, Theorem 3.1]. □

Recall that a ring R is called Gaussian if $c(fg) = c(f)c(g)$ for every polynomials $f, g \in R[X]$, where $c(f)$ is the content of f, that is, the ideal of R generated by the coefficients of f. See for instance [8].

Corollary 2.6. *Let R be a reduced ring and let I be a pure ideal of R. Then R is Gaussian if, and only if, $R \bowtie I$ is Gaussian.*

Proof. Follows immediately from Theorem 2.1 , [8, Theorem 2.2] and [6, Theorem 3.5(a)(vi)]. □

By the fact that every ideal over a Von Neumann regular ring is pure, we conclude from Theorem 2.1 the following Corollary which have already proved in [3] with different methods.

Corollary 2.7. *Let R be a ring and let I be an ideal of R. If R is a Von Neumann regular ring, then so is $R \bowtie I$.*

If the ring R is Noetherian the global and weak dimensions coincide. Hence, Theorem 2.1 can be writing as follows:

Corollary 2.8. *If I is a pure ideal of a Noetherian ring R, then $gdim(R \bowtie I) = gdim(R)$.*

A simple example of Theorem 2.1 is given by introducing the notion of the trace of modules. Recall that if M is an R-module, the trace of M, $Tr(M)$, is the sum of all images of morphisms $M \rightarrow R_R$ (see [15]). Clearly $Tr(M)$ is an ideal of R.

Example 2.9. If M is a projective module over a ring R, then $wdim(R \bowtie Tr(M)) = wdim(R)$.

Proof. Clear since $Tr(M)$ is a pure ideal whenever M is projective (by [19, pp. 269-270]). □

Now, we study the transfer of an $(1, d)$-property.

Theorem 2.10. *Let R be a coherent ring such that for every maximal ideal m of R the local ring R_m is an $(1, d)$-ring, and let I be a finitely generated pure ideal of R. Then $R \bowtie I$ is an $(1, d)$-ring.*

Proof. Using [1, Theorem 3.2] and [3, Theorem 3.1], we have to prove that for any maximal ideal M of $R \bowtie I$, the ring $(R \bowtie I)_M$ is an $(1, d)$-ring. So, let M be such ideal and set $m := M \cap R$. From the proof of Theorem 2.1, we have two possible cases:

(1) $(R \bowtie I)_M \cong R_m$ if $I_m = 0$ or $I \nsubseteq m$.
(2) $(R \bowtie I)_M \cong R_m \times R_m$ if $I_m = R_m$ or $I \subseteq m$.

So, by the hypothesis conditions, $(R \bowtie I)_M$ is an $(1, d)$-ring since R_m is it, as desired. □

By the fact that every ideal over a semisimple ring is pure we conclude from Theorem 2.10 the following Corollary.

Corollary 2.11. *Let R be a ring and let I be an ideal of R. If R is a semisimple ring, then so is $R \bowtie I$.*

Now, we give a wide class of rings which are not weak (n, d)-rings (and so not (n, d)-rings) for each positive integers n and d.

Theorem 2.12. *Let R be a ring and let I be a proper ideal of R which satisfies the following condition:*

(1) R_m is a domain for every maximal ideal m of R.
(2) I_m is a principal proper ideal of R_m for every maximal ideal m of R.

Then, $wdim(R \bowtie I)(= gldim(R \bowtie I)) = \infty$.

Proof. Let m be a maximal ideal of R such that $I \subseteq m \subsetneq R$. By Lemma 2.3, $R_m \bowtie I_m = (R \bowtie I)_M$ where $M = \{(p, p+i) | p \in m, i \in I\}$. From [3, Theorem 2.13] and by the hypothesis conditions, we have $wdim(R \bowtie I)_M = wdim(R_m \bowtie I_m) = \infty$. Then, the desired result follows from [7, Theorem 1.3.14]. $\square$

The following example shows that the condition "I_m is a principal proper ideal of R_m for every maximal ideal m in R" is necessary in Theorem 2.12.

Example 2.13. Let R be a Von Neumann regular ring and let I be a proper ideal of R. Then $wdim(R \bowtie I) = 0$ since $(R \bowtie I)$ is a Von Neumann regular ring, and I_m is not a proper ideal of R_m since R_m is a field.

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